

Problem 1.14

[Difficulty: 4]

1.14 A sky diver with a mass of 70 kg jumps from an aircraft. The aerodynamic drag force acting on the sky diver is known to be $F_D = kV^2$, where $k = 0.25 \text{ N}\cdot\text{s}^2/\text{m}^2$. Determine the maximum speed of free fall for the sky diver and the speed reached after 100 m of fall. Plot the speed of the sky diver as a function of time and as a function of distance fallen.

Given: Data on sky diver: $M = 70 \cdot \text{kg}$ $k = 0.25 \cdot \frac{\text{N}\cdot\text{s}^2}{\text{m}^2}$

Find: Maximum speed; speed after 100 m; plot speed as function of time and distance.

Solution: Use given data; integrate equation of motion by separating variables.

Treat the sky diver as a system; apply Newton's 2nd law:

Newton's 2nd law for the sky diver (mass M) is (ignoring buoyancy effects):

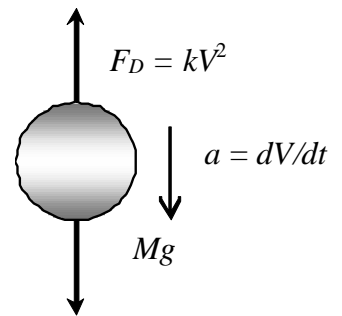
$$M \cdot \frac{dV}{dt} = M \cdot g - k \cdot V^2 \quad (1)$$

(a) For terminal speed V_t , acceleration is zero, so $M \cdot g - k \cdot V^2 = 0$ so

$$V_t = \sqrt{\frac{M \cdot g}{k}}$$

$$V_t = \left(70 \cdot \text{kg} \times 9.81 \cdot \frac{\text{m}}{\text{s}^2} \times \frac{\text{m}^2}{0.25 \cdot \text{N}\cdot\text{s}^2} \cdot \frac{\text{N}\cdot\text{s}^2}{\text{kg} \times \text{m}} \right)^{\frac{1}{2}}$$

$$V_t = 52.4 \frac{\text{m}}{\text{s}}$$



(b) For V at $y = 100 \text{ m}$ we need to find $V(y)$. From (1) $M \cdot \frac{dV}{dt} = M \cdot \frac{dV}{dy} \cdot \frac{dy}{dt} = M \cdot V \cdot \frac{dV}{dt} = M \cdot g - k \cdot V^2$

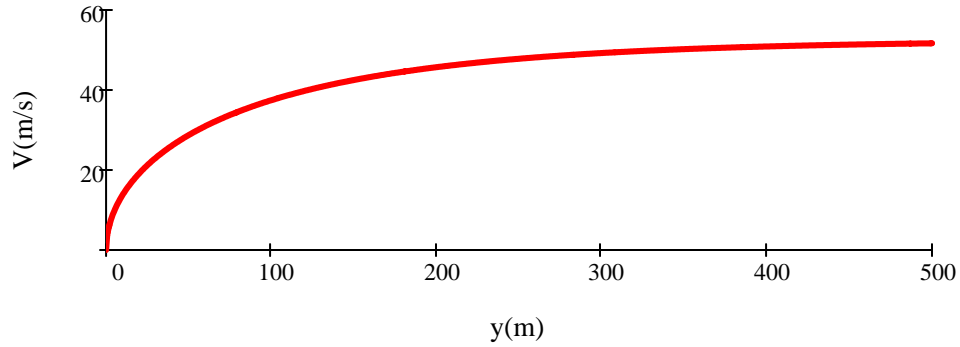
Separating variables and integrating:

$$\int_0^V \frac{V}{1 - \frac{k \cdot V^2}{M \cdot g}} dV = \int_0^y g dy$$

so $\ln \left(1 - \frac{k \cdot V^2}{M \cdot g} \right) = -\frac{2 \cdot k}{M} y$ or $V^2 = \frac{M \cdot g}{k} \cdot \left(1 - e^{-\frac{2 \cdot k \cdot y}{M}} \right)$

Hence $V(y) = V_t \cdot \left(1 - e^{-\frac{2 \cdot k \cdot y}{M}} \right)^{\frac{1}{2}}$

For $y = 100 \text{ m}$: $V(100 \cdot \text{m}) = 52.4 \cdot \frac{\text{m}}{\text{s}} \cdot \left(1 - e^{-2 \times 0.25 \cdot \frac{\text{N}\cdot\text{s}^2}{\text{m}^2} \times 100 \cdot \text{m} \times \frac{1}{70 \cdot \text{kg}} \times \frac{\text{kg} \cdot \text{m}}{\text{s}^2 \cdot \text{N}}} \right)^{\frac{1}{2}}$ $V(100 \cdot \text{m}) = 37.4 \cdot \frac{\text{m}}{\text{s}}$



(c) For $V(t)$ we need to integrate (1) with respect to t :

$$M \cdot \frac{dV}{dt} = M \cdot g - k \cdot V^2$$

Separating variables and integrating:

$$\int_0^V \frac{V}{\frac{M \cdot g}{k} - V^2} dV = \int_0^t 1 dt$$

so

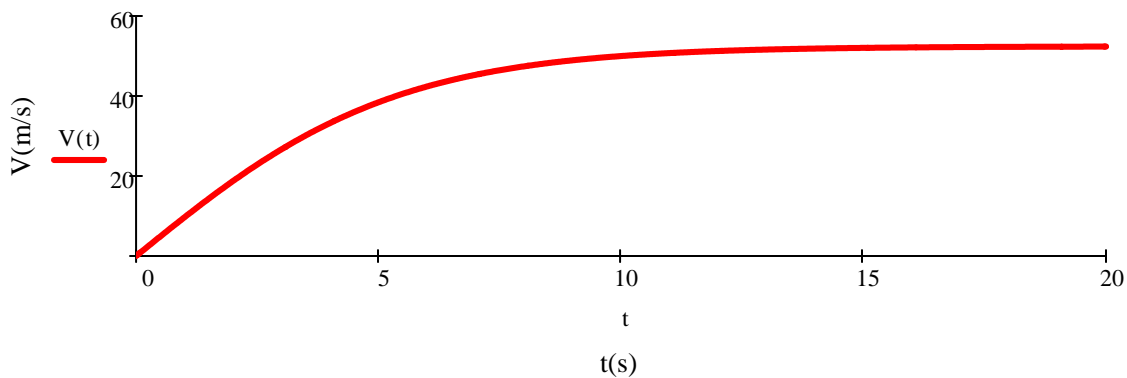
$$t = \frac{1}{2} \cdot \sqrt{\frac{M}{k \cdot g}} \cdot \ln \left(\frac{\sqrt{\frac{M \cdot g}{k}} + V}{\sqrt{\frac{M \cdot g}{k}} - V} \right) = \frac{1}{2} \cdot \sqrt{\frac{M}{k \cdot g}} \cdot \ln \left(\frac{|V_t + V|}{|V_t - V|} \right)$$

Rearranging

$$V(t) = V_t \cdot \frac{\left(e^{\frac{2 \cdot \sqrt{\frac{k \cdot g}{M}} \cdot t}{2}} - 1 \right)}{\left(e^{\frac{2 \cdot \sqrt{\frac{k \cdot g}{M}} \cdot t}{2}} + 1 \right)}$$

or

$$V(t) = V_t \cdot \tanh \left(V_t \cdot \frac{k}{M} \cdot t \right)$$



The two graphs can also be plotted in Excel.